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STABILITY AND BRANCHING OF NORMAL OSCILLATION FORMS OF NONLINEAR SYSTEMS*

A.L. ZHUPIEV and Iu.V. MIKHLIN

Essentially nonlinear conservative systems that admit single-frequency modes, i.e. normal oscillations with rectilinear trajectories are considered. In many cases the passage to consideration of trajectories in the configuration space makes possible a simplification of investigation of normal oscillation orbital stability, a problem which reduces to four Sturm-Liouville problems with equations in variations. All points of stability change of normal oscillations, which are also branching points, are determined for homogeneous systems. Various types of periodic modes of branching from normal form solutions (whether affecting or not affecting normal oscillations stability) are analyzed. The effect of mechanical characteristics of a nonlinear chain system with two degrees of freedom on the number of normal oscillation forms and their stability is studied. The obtained results are applied to nearly-homogeneous systems.

1. Let us consider a conservative system with n degrees of freedom admitting oscillations of rectilinear form

$$Mu^{\cdot\cdot} + dV/du = 0 \tag{1.1}$$

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),\ u = (u_1, u_2, \ldots, u_n)^T,\quad \frac{d}{du} = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_n}\right)^T$$

where $V = V(u)$ is the system potential which is an even analytic function. The rectilinear normal forms of the system are determined by the relations

$$u = Cx(t),\quad C = (C_1, C_2 \ldots, C_n)^T$$

in which the constants C_i can be obtained using algebraic equations /1/, and function $x(t)$ from the equation

$$x^{\cdot\cdot} + (C, dV(x)/du) = 0$$

where the normalization condition $(C, MC) = 1$ was used.

In investigating the stability of normal oscillations $u = Cx(t)$ we use the respective equations in variations

$$M\delta u^{\cdot\cdot} + (d^2V(Cx)/du^2)\delta u = 0$$

whose solution will be sought in the form of functions of the variable x. In this system of coordinates normal oscillations are defined by straight lines, which in a number of cases facilitates the investigation of stability. The passing to such coordinate system may be termed the geometrization of the stability problem.

We restrict the investigation to systems that admit diagonalization of matrix $d^2V(x)/du^2$ by means of some transform $y = P\delta u$, where P is a constant nonsingular matrix. The theorem on the possibility of such diagonalization was proved earlier in /2/. It is feasible, when a conservative system which admits rectilinear oscillations of normal form and has two degrees of freedom, or when a system with any number of degrees of freedom is in the class of homogeneous, symmetric, or other systems.

The system of equations in variations decomposes now into n independent equations, one of which defines variations along the rectilinear trajectory, the remaining in directions orthogonal to it. Orbital stability of normal oscillations is linked with the latter variations. It should be pointed out that each equation is of one and the same form

$$2Wy'' + W'y' + Gy = 0,\quad W = (1 - x^2)\overline{W} \tag{1.2}$$

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where $\overline{W}$ is an analytic function which has no real roots. Here, without loss of generality, the scale has been selected for convenience so that the amplitude of the generating solution is equal unity.

The Floquet—Liapunov theory shows that the stability and instability domains of solutions of equations in variations in the system parameter space are separated by T- and $T/2$-periodic solutions which branch from the generating solution /3/, with the domains of dynamic instability of the form of small wedges that contract to points when deviations from the generating solution approach zero /4/.

Let G be a function of a certain number of parameters. The problem of determining the T- and $T/2$-periodic solutions decomposes in four Sturm—Liouville problems in even and odd functions regular at points $x = \pm 1$, or having singularities of the form $\sqrt{1-x^2}$ at those points.

If the system potential is a polynomial, each equation in variations is a generalized Lamé equation, while the eigenfunctions of the Sturm—Liouville problems are generalized Lamé functions.

The periodic solutions that branch off from normal oscillations at stability change, merge again at the change of system parameters with some normal oscillations, but without affecting the stability of these oscillations. The problem of deriving such solutions becomes a problem in eigenvalues $S^m = I$, where S is the matrix of the analytic continuation of the general solution over the closed contour that includes the singular points $x = \pm 1$.

2. Certain problems of dynamics of multilayer structures, as well as those with shrouded components reduce to the investigation of homogeneous systems that admit oscillations of rectilinear form, and to systems close to homogeneous. In the case of homogeneous systems whose potential is a homogeneous function of order p (linear systems are such) each equation in variations (1.2) is reduced by the substitution $x^p = z$ to a hypergeometric equation of the form

$$y''z(1-z) + \left(\frac{p-1}{p} - \frac{3p-2}{2p}z\right)y' + \frac{\lambda}{2p}y = 0 \qquad (2.1)$$

In this case the problem of stability and branching, formulated above for equations in variations, is considerably simplified. It reduces to the determination of such parameter λ for which, after passing over the closed contour containing the singular points $z = 0$, $z = 1$, the solution is multiplied by $+1$ or -1. Such solutions are called degenerate. They are given in /5/, and are of the form

$$y = z^{\mu_1}(1-z)^{\mu_2} r_n(z)$$

where $r_n(z)$ is a polynomial, and μ_1 can be equal zero or $1/p$ and μ_2 either zero or $^1/_2$.

By analogy to the Mathieu equation we denote the eigenfunctions which are Gegenbauer's polynomials by $C_{4k}(z)$ (an even $T/2$-periodic solution), by $C_{4k+1}(z)$ (an odd T-periodic solution), by $S_{4k+2}(z)$ (an even T periodic solution), and by $S_{4k+3}(z)$ (an odd $T/2$- periodic solution). In the linear case the trajectories of these solutions are of the form of Lissajou figures. Below we present in the same order the eigenvalues of these problems

$$\lambda_{4k} = k(2kp + p - 2), \quad \lambda_{4k+1} = (2k+1)(kp+1)$$
$$\lambda_{4k+2} = (2k+1)(kp+p-1), \quad \lambda_{4k+3} = (k+1)(2kp+p+2)$$
$$(k = 0, 1, 2, \ldots)$$

As an example, let us consider the homogeneous chain system with two degrees of freedom

$$m_1x_1^{\cdot\cdot} + C_{11}x_1^{p-1} + C_{12}(x_1 - x_2)^{p-1} = 0$$
$$m_2x_2^{\cdot\cdot} + c_{22}x_2^{p-1} - c_{12}(x_1 - x_2)^{p-1} = 0$$

Equations for the determination of the oscillation form are

$$\varepsilon = \frac{\alpha\mu^{p-1} - \varkappa\mu}{(1-\mu)^{p-1}(1+\varkappa\mu)} \quad \left(\alpha = \frac{c_{22}}{c_{11}},\ \varepsilon = \frac{c_{12}}{c_{11}},\ \varkappa = \frac{m_2}{m_1},\quad \mu = \frac{x_2}{x_1}\right)$$

Analysis of this equation shows that for positive values of the constraint parameter ε a single cophasal oscillation form always exists and, depending on ε and the homogeneity index p, there are one, three, or five antiphased forms. For a negative constraints parameter ε there exist for any p either a single antiphased, or one or three cophased oscillation forms.

In this example the eigenvalues of the hypergeometric equation in variation are of the form

$$\lambda = \frac{(p-1)(1+\varkappa\mu)(\alpha\mu^{p-2}-\varkappa\mu)}{\varkappa(1-\mu)(\alpha\mu^{p-1}+1)}$$

The curves of relation between $\eta = 4\pi^{-1}\,\mathrm{arctg}\,(\varepsilon\cdot 2^{p-1}/(p-2))$ and $\xi = \mathrm{arctg}\,\mu$ (Fig.1) enable us to establish the dependence of the number of forms on parameters of the chain system. Curve a corresponds to $p = 4$, $\alpha = 1$; b is $p = 8$, $\alpha = 1$; c is $p = 4$, $\alpha = 0$; d is $p = 4$, $\alpha = 1, 2$ (with $\varkappa = 1$ everywhere). The first points of stability change for $p = 4$ are indicated by the symbol 0 and for $p = 8$ by the symbol $\times$, with the numeral near a point corresponding to the number of the eigenvalue λ_i. Sections of instability of oscillation forms lie between points with numerals from $4k+1$ to $4k+2$ and from $4k+3$ to $4k+4$ $(k = 0, 1, 2, \ldots)$. In the asymmetric case $(\alpha \neq 1)$ some of the branching points become limit points.

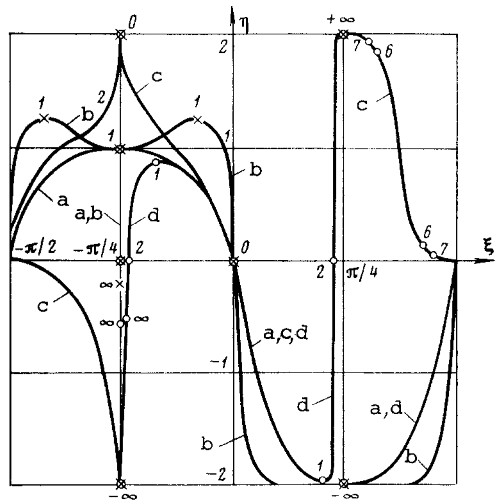

Fig.1

To determine branching points of periodic solutions, which do not affect stability of normal oscillations we use the general solution of the equation in variations (2.1) in the form

$$y = C_1 u_1(z) + C_2 z^{1/(p-1)} u_5(z) \quad \text{(near the singular point } z = 0)$$
$$y = D_1 u_2(1-z) + D_2\sqrt{1-z}\, u_6(1-z) \quad \text{(near the singular point } z = 1)$$

where C_1, C_2, D_1, D_2 are aribtrary constants and $u_1(z)$, $u_2(1-z)$, $u_5(z)$, $u_6(1-z)$ are known hypergeometric functions /5/. We denote by N matrix

$$\left\| \begin{matrix} C_1 \\ C_2 \end{matrix} \right\| = N \left\| \begin{matrix} D_1 \\ D_2 \end{matrix} \right\|$$

which connects constants C_1, C_2 and D_1, D_2 .

It is possible to formulate a countable number of boundary value problems

$$(JN^{-1}JN)^m = I \quad (m = 1, 2, \ldots); \quad J = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|$$

for the determination of branching periodic solutions that pass m times round the singular points $z = 0$ and $z = 1$.

For the eigenvalue $\lambda = (p-2)/(p-1)$ such periodic solution can be written in the explicit form

$$y = C_1(1 + \sqrt{1 - z^{p-1}})^{1/(p-1)} + C_2 z (1 + \sqrt{1 - z^{p-1}})^{-1/(p-1)}$$

3. The obtained results can be used in the case of systems close to periodic. Excluding from the investigation those values of parameter λ for which a change of stability of the generating normal oscillations takes place, it is always possible to construct near-periodic

solutions, i.e. obtain normal oscillations with curvilinear trajectories /6/. Periodic solutions of equations in variations which separate the domains of stability and instability are to be sought by the method of perturbations. In the zero approximation these solutions are defined by eigenfunctions of the hypergeometric equation that correspond to the generating homogeneous system. In the following approximations we determine the points of stability change using the conditions of periodicity of solutions.

As an example we consider a symmetric chain system whose potential contains terms with second and fourth powers of variables x_1 and x_2

$$\begin{aligned} &mx_1^{\cdot\cdot} + c_{11}x_1 + c_{13}x_1^3 + c_{21}(x_1 - x_2) + c_{23}(x_1 - x_2)^3 = 0 \\ &mx_2^{\cdot\cdot} + c_{11}x_2 + c_{23}x_2^3 - c_{21}(x_1 - x_2) - c_{23}(x_1 - x_2)^3 = 0 \end{aligned} \tag{3.1}$$

The branching point of rectilinear oscillation forms, which determines the first stability change of such periodic solutions was isolated in /7/ in the case of similar systems with a single parameter.

The equation which defines the antiphased oscillation form, and the equation in variations which makes possible the assessment of that phase orbital stability are of the form

$$u^{\cdot\cdot} + u(\gamma + 2\rho u^2) = 0,\ v^{\cdot\cdot} + v(\sigma + \beta u^2) = 0$$

$$\gamma = \frac{c_{11} + 2c_{21}}{m},\quad \rho = \frac{c_{13} + 8c_{23}}{m},\quad \sigma = \frac{c_{11}}{m},\quad \beta = \frac{3c_{13}}{m}$$

$$u = \frac{x_1 - x_2}{2},\quad v = \frac{\delta x_1 + \delta x_2}{2}$$

or (the prime denotes differentiation with respect to u)

$$v''(\gamma(1 - u^2) + \rho(1 - u^4)) - v'(\gamma u + 2\rho u^3) + (\sigma + \beta u^2) = 0 \tag{3.2}$$

which is the Lamé equation that for specific relations between parameters $\sigma, \beta, \gamma, \rho$ has solutions in the form of Lamé polynomials or functions. When system (3.1) is nearly linear or cubic, the indicated above perturbation method for determining the first two points of stability change (branching of normal oscillations, and of periodic oscillations with the period of the generating solution, but shifted in phase by a half-cycle) yields the following expressions:

$$\gamma_0 = \sigma_0,\quad \beta_0 = \rho_0 = 0 \quad \text{or} \quad \gamma_0 = \sigma_0 = 0,\quad \beta_0 = 2\rho_0$$

$$\frac{\gamma_1 - \sigma_1}{\beta_1 - 2\rho_1} = 0.75 \quad \text{or} \quad \frac{\gamma_1 - \sigma_1}{\beta_1 - 2\rho_1} = \frac{1}{12}\left[\frac{\Gamma(^1/_4)}{\Gamma(^3/_4)}\right]^2 \approx 0.729$$

for quasi-linear and cubic systems, respectively, for the first point of stability change and

$$\gamma_0 = \sigma_0,\quad \beta_0 = \rho_0 = 0 \quad \text{or} \quad \gamma_0 = \sigma_0 = 0,\quad \beta_0 = 6\rho_0$$

$$\frac{\gamma_1 - \sigma_1}{\beta_1 - 6\rho_1} = 0.25 \quad \text{or} \quad \frac{\gamma_1 - \sigma_1}{\beta_1 - 6\rho_1} = \frac{12}{5}\left[\frac{\Gamma(^3/_4)}{\Gamma(^1/_4)}\right]^2 \approx 0.274$$

for the second point of stability change.

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